Two remarks on non-zero constant Jacobian polynomial map of $\mathbb{C}^{2*}$

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Abstract

We present some estimations on geometry of the exceptional value sets of non-zero constant Jacobian polynomial maps of $\mathbb{C}^2$ and it's components.

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1. Introduction. Recall that the *exceptional value set* E_h of a polynomial mapping $h : \mathbb{C}^m \longrightarrow \mathbb{C}^n$ is the smallest subset $E_h \subset \mathbb{C}^n$ such that the restriction $h : h^{-1}(\mathbb{C}^n \setminus E_h) \longrightarrow \mathbb{C}^n \setminus E_h$ gives a locally trivial smooth fibration. The mysterious Jacobian conjecture (See [BMW] and [E]), which posed first by Keller in 1939 and still open even for two dimensional case, asserts that a polynomial map of $\mathbb{C}^n$ with non-zero constant Jacobian must be a polynomial bijection. Consider a polynomial map $f = (P, Q) : \mathbb{C}^2_{(x,y)} \longrightarrow \mathbb{C}^2_{(u,v)}$ and denote $J(P,Q) := P_x Q_y - P_y Q_x$. It is well-known that if $J(P,Q) \equiv const. \neq 0$ but f is not bijective, then the sets E_P and E_Q must be non empty finite sets and E_f is a curve so that each of it's irreducible components is a polynomially parameterized curve (the image of a polynomial map from $\mathbb{C}$ into $\mathbb{C}^2$) (See, for example in [J1, J2]).

In this note we present the followings.

Theorem 1. *Suppose $f = (P, Q)$ is a polynomial map with non-zero constant Jacobian. Then, a value $u_0 \in E_P$ if and only if the line $u = u_0$ tangents to an irreducible local branch of E_f.*

This theorem leads to the following fact that may be used to consider the non-existence of non-zero constant Jacobian polynomial maps with exceptional value curve of given types.

Theorem 2. *The exceptional value set of a polynomial map of $\mathbb{C}^2$ with non-zero constant Jacobian can not be isomorphic to a curve composed of the images of some polynomial maps of the form $t \mapsto (t^k, q(t))$, $k \in \mathbf{N}$, $q(t) \in \mathbb{C}[t]$.*

A special property of the curve with irreducible components given by parameterizations in theorem 2 is that it's local irreducible branches may tangent to the only line $u = 0$. Simply connected curves are simple examples of such curves. This can be easy deduced from Lin-Zaidenberg's theorem on simply connected curves (Theorem B in [ZL]).

Corollary 3. *The exceptional value set of a polynomial map of $\mathbb{C}^2$ with non-zero constant Jacobian can not be a simply connected curve.*

Proofs of Theorem 1 and Theorem 2 will be presented in $ 3 and $4.

2. Preliminaries. In this section, we present some elementary facts which will be used in the proof of Theorem 1.

i) We will work with finite factional series with parameter ξ of the form

$$\varphi(x, \xi) = \sum_{k=1}^{K-1} a_k x^{\frac{n_k}{m}} + \xi x^{\frac{n_K}{m}}, 0 \neq a_k \in \mathbb{C}, n_k \in \mathbf{Z}, \ m \in \mathbf{N} \qquad (1)$$

with $n_1 > n_2 > \ldots > n_K$ and $\gcd(\{n_k : k \leq K\}) = 1$.

Let $h(x, y)$ be a non-constant primitive polynomial monic in y, $\deg_y h = \deg h$. A fractional series $\varphi(x, \xi)$ in (1) is called *a Newton-Puiseux type* of h if

$$h(x, \varphi(x, \xi)) = h_\varphi(\xi) + \text{lower terms in } x, \ h_\varphi \in \mathbb{C}[\xi], \ \deg h_\varphi > 0.$$

Denote by $\mathrm{mult}(\varphi) := m$ and $\mathrm{ord}(\varphi) := n_K$.

The Newton-Puiseux types of $h(x,y)$ can be constructed from Newton-Puiseux expansions at infinity of the curve $h(x,y) = 0$. In fact, if $y = y(x)$ is such a Newton-Puiseux expansion at infinity, then there is a unique Newton-Puiseux type φ of h and a number $\xi_0 \in \mathbb{C}$ such that $y(x) = \varphi(x,\xi_0) +$ lower terms in x.

For a Newton-Puiseux type φ of h the rational map $\Phi : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}\,\mathbb{P}^2$ defined by

$$\Phi(t,\xi) := (t^{-\mathrm{mult}(\varphi)}, \varphi(t^{-\mathrm{mult}(\varphi)}, \xi)). \tag{2}$$

determines a unbranched i_φ sheeted covering from $\mathbb{C}^* \times \mathbb{C}$ onto $\Phi(\mathbb{C}^* \times \mathbb{C}) \subset \mathbb{C}^2$, where $i_\varphi := \mathrm{mult}(\varphi)/\gcd\{k : a_k \neq 0,\ k < K\}$. Hence, one can use the polynomial $H_\varphi(t,\xi) := h \circ \Phi(t,\xi)$ as a kind of "extension" of $h(x,y)$.

Lemma 4. (See Theorem 2.4, [C]). *Suppose $h(x,y)$ is a primitive polynomial and monic in y. Then, $c \in E_h$ if and only if c is either a critical value of h or a critical value of $h_\varphi(\xi)$ for any Newton-Puiseux type φ of h.*

In fact, by Newton's theorem the polynomial $h(x,y) - c$ can be factorized as

$$h(x,y) - c = C \prod_u (y - u(x)), \tag{3}$$

where the product runs over all Newton-Puiseux expansion at infinity of the curve $h = c$. Substituting $y = \varphi(x,\xi)$ into this representation, one can see that a number $\alpha \in \mathbb{C}$ is a zero point of $h_\varphi(\xi) - c = 0$ with multiplicity n if and only if the level $h = c$ has exactly n Newton-Puiseux expansions at infinity of the form $\varphi(x,\alpha) +$ lower term in x. Thus, Puiseux data at infinity of the curves $h = c$ must be changed when c is a critical values of h_φ.

ii) Consider a polynomial map $f = (P,Q) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$. By definition the exceptional value set E_f is composed by the critical value set of f and the so-called *non-proper value set* A_f of f - the set of all value $a \in \mathbb{C}^2$ such that there exists a sequence $\mathbb{C}^2 \ni p_i \to \infty$ with $f(p_i) \to a$. Following [J1], the non-proper value set A_f, if non-empty, is a curve composed by the images of some polynomial maps from $\mathbb{C}$ into $\mathbb{C}^2$.

A series $\varphi(x,\xi)$ in (1) is called *dicritical series* of f if

$$f(x, \varphi(x,\xi)) = f_\varphi(\xi) + \text{lower terms in } x,\ \deg f_\varphi > 0.$$

Lemma 5.

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$$A_f = \bigcup_{\varphi \text{ is a dicritical series of } f} f_\varphi(\mathbb{C}).$$

Proof: Let φ be a dicritical series of f and $\Phi(t,\xi)$ be as in (2). The map Φ sends $\mathbb{C}^* \times \mathbb{C}$ to $\mathbb{C}^2$ and the line $\{0\} \times \mathbb{C}$ to the line at infinity of $\mathbb{C}\,\mathbb{P}^2$. Then, the polynomial map $F_\varphi(t,\xi) := f \circ \Phi(t,\xi)$ maps the line $\{0\} \times \mathbb{C}$ into $A_f \subset \mathbb{C}^2$. Therefore, $f_\varphi(\mathbb{C})$ is an irreducible component of A_f, since $\deg f_\varphi > 0$.

Conversely, assume that ℓ is an irreducible component of A_f. By definition we can choose a smooth point (u_0, v_0) of ℓ and an irreducible branch at infinity γ of the curve $P = u_0$ (or the curve $Q = v_0$) such that the image $f(\gamma)$ is a branch curve intersecting transversally ℓ at (u_0, v_0). Let $\varphi(x, \xi)$ be the Newton-Puiseux type of P constructed corresponding to a Newton-Puiseux expansion at infinity of γ. Then, by definitions we can verify that φ is a dicritical series of f and $f_\varphi(\mathbb{C}) = \ell$. Q.E.D

3. Proof of Theorem 1. We consider a polynomial map $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ with non-zero constant Jacobian. Fix a suitable affine coordinate (x, y) so that P and Q is monic in y. For a series φ in (1) represent

$$P(x, \varphi(x, \xi)) = p_\varphi(\xi) x^{\frac{a_\varphi}{\text{mult}(\varphi)}} + \text{lower terms in } x$$
$$Q(x, \varphi(x, \xi)) = q_\varphi(\xi) x^{\frac{b_\varphi}{\text{mult}(\varphi)}} + \text{lower terms in } x,$$

Lemma 6. *i) Let φ be a Newton-Puiseux type of P. If φ is not a dicritical series of f, then $\deg p_\varphi(\xi) = 1$, $q_\varphi(\xi) \equiv \text{const.} \neq 0$ and $b_\varphi > 0$.*

ii) A dicritical series of f must be a Newton-Puiseux type both of P and Q.

Proof. i) First, we will show that

$$\deg p_\varphi = 1, \ q_\varphi \equiv const \neq 0. \tag{$*$}$$

Taking differentiation of $F(t^{-\text{mult}(\varphi)}, \varphi(t^{-\text{mult}(\varphi)}, \xi))$, as $a_\varphi = 0$ and $b_\varphi \neq 0$ we have

$$\text{mult}(\varphi) J(P, Q) t^{\text{ord}(\varphi)-\text{mult}(\varphi)-1} = -b_\varphi \dot{p}_\varphi q_\varphi t^{-b_\varphi - 1} + \text{higher terms in } t.$$

Since $J(P, Q) \equiv const. \neq 0$ and $\deg p_\varphi > 0$, we get ($*$).

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Now, assume the contrary that $b_\varphi < 0$. Then, there exists a Newton-Puiseux root at infinity $u(x)$ of the curve $Q = 0$ such that $u(x) = \varphi(x, \xi_0) +$ lower term in x. Let $\psi(x, \xi)$ be the Newton-Puiseux type of Q constructed corresponding to $u(x)$. Obviously, $a_\psi > a_\varphi = 0$ and hence ψ is not a dicritical series of f. Furthermore, $\varphi(x, \xi) = \psi(x, \alpha) +$ lower terms in x for a zero point α of $p_\psi(\xi)$. This is impossible, since $p_\psi(\xi) \equiv const. \neq 0$ by applying (*) to the Newton-Puiseux type ψ of Q. Thus, we get $b_\varphi > 0$.

ii) This is obtained from (i) and definitions. Q.E.D

Proof of Theorem 1. If $E_f = \emptyset$, then f is bijective and $E_P = E_Q = \emptyset$. Hence, we need consider only the situation when $E_f \neq \emptyset$. In this situation $E_f = A_f$, since f has not singularity.

First, suppose the line $u = u_0$ tangents to a local irreducible branch curve of an irreducible component ℓ of A_f. By Lemma 5, there is a dicritical series φ of f such that ℓ is the image of $f_\varphi := (p_\varphi, q_\varphi)$. By Lemma 6 (ii) the series φ is a Newton-Puiseux type both of P and Q. Since the line $u = u_0$ tangents to ℓ, u_0 must be a critical value of p_φ. Hence, by Lemma 4 the number u_0 is an exceptional value of P.

Conversely, let $u_0 \in \mathbb{C}$ and denote $L := \{u = u_0\}$. Assume that L intersects transversally each of local irreducible branches of A_f located at points in $L \cap A_f$. We have to show that u_0 can not be an exceptional value of P. In view of Lemma 4 and Lemma 6 we need only to verify that u_0 is a regular value of $p_\varphi(\xi)$ for every dicritical series φ of f.

Let φ be a given dicritical series of f and $\ell := f_\varphi(\mathbb{C})$, which is an irreducible component of A_f by Lemma 5. Let $(u_0, v_0) \in \ell$ and $\xi_0 \in \mathbb{C}$ with $f_\varphi(\xi_0) = (u_0, v_0)$. We have to show that $\dot{p}_\varphi(\xi_0) \neq 0$.

Consider the polynomial map $F : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}^2$

$$F(t, \xi) := f \circ (t^{-\operatorname{mult}(\varphi)}, \varphi(t^{-\operatorname{mult}(\varphi)}, \xi)) = (p_\varphi, q_\varphi)(\xi) + \text{ higher terms in } t.$$

For this map $F(\{0\} \times \mathbb{C}) = \ell$ and

$$\det DF(t, \xi) = -\operatorname{mult}(\varphi) J(P, Q) t^{\operatorname{ord}(\varphi) - \operatorname{mult}(\varphi) - 1}.$$

Since $J(P, Q) \equiv const. \neq 0$, F has singularity only on the line $t = 0$.

Let $\gamma := f_\varphi(\{\xi : |\xi - \xi_0| < \epsilon\}$ for an enough small $\epsilon > 0$. As assumed the line L intersects transversally γ at (u_0, v_0). So, we can choose an enough small neighborhood U of (u_0, v_0) so that $\gamma := \ell \cap U$ is a smooth branch

curve parameterized by $v = v_0 + h(u - u_0)$ for a holomorphic function h, $h(0) = 0$. Define new coordinates $(\bar{u}, \bar{v}) = (u - u_0, v - v_0 - h(u - u_0))$ in U and $(\bar{t}, \bar{\xi}) = (t, \xi - \xi_0)$ in an enough small neighborhood V of $(0, \xi_0)$. Let $\bar{F} = (\bar{F}_1, \bar{F}_2)$ is the representation of F in these coordinates. Then,

$$\bar{F}_1(\bar{t}, \bar{\xi}) = p_\varphi(\bar{\xi}) - u_0 + \text{ higher terms in } \bar{t}, \tag{4}$$

$\bar{F}(0,0) = (0,0)$, $\bar{F}(\{\bar{t} = 0\}) \subset \gamma = \{\bar{v} = 0\}$ and $\det D\bar{F}(\bar{t}, \bar{\xi}) \neq 0$ for $\bar{t} \neq 0$. Then, by examining Newton diagrams of $\bar{F}_1$, $\bar{F}_2$ and $\det D\bar{F}$ we can verify that

$$\bar{F}(\bar{t}, \bar{\xi}) = (\bar{\xi}u_1(\bar{t}, \bar{\xi}) + \bar{t}u_2(\bar{t}, \bar{\xi}), \bar{t}^k v_1(\bar{t}, \bar{\xi})), \tag{5}$$

where u_1, u_2 and v_1 are holomorphic functions define in V, $u_1(0, 0) \neq 0$ and $v_1(0, 0) \neq 0$ (See, for example [O, Lemma 4.1]). From (4) and (5) it follows that $\dot{p}_\varphi(\xi_0) \neq 0$. Q.E.D

Remark. From Lemma 6 (ii) and Lemma 5 one can easy see that the exceptional value set E_f of nonsingular polynomial map f can not contains an irreducible component isomorphic to a line.

4. Proof of Theorem 2. Let $f = (P, Q)$ be representation of f in the coordinate in which E_f consists with the images of some polynomial maps of the form $t \mapsto (t^k, q(t))$, $k \in \mathbf{N}$. By applying Theorem 1 we have that $E_P \subset \{0\}$. As f has not singularity, P is a nonsingular primitive polynomial. Then, from Suzuki's equality

$$\sum_{c \in \mathbb{C}} (\chi_c - \chi) = 1 - \chi$$

(See in [S]) we get $\chi_0 = 1$. Here, χ_c and χ indicate the Euler-Poincare characteristic of the fiber $P = c$ and the generic fiber of P, respectively. Since the curve $P = 0$ is smooth, it has one connected component ℓ diffeomorphic to $\mathbb{C}$. This component ℓ must be isomorphic to $\mathbb{C}$ by the Abhyankar-Moh Theorem [AM] and the restriction of f on ℓ must be injective. Then, as observed by Gwozdziewicz in [G], f must be bijective. This is impossible, since $E_f \neq \emptyset$.Q.E.D

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